

CONSTRUCTION OF SEPARATELY CONTINUOUS FUNCTIONS WITH GIVEN RESTRICTION

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ABSTRACT. It is solved the problem on constructed of separately continuous functions on product of two topological spaces with given restriction. In particular, it is shown that for every topological space X and first Baire class function $g : X \rightarrow \mathbf{R}$ there exists separately continuous function $f : X \times X \rightarrow \mathbf{R}$ such that $f(x, x) = g(x)$ for every $x \in X$.

1. INTRODUCTION

It is well-known (see [1]) that the diagonals of separately continuous functions of two real variables are exactly the first Baire functions. It is shown in [2] that for every topological space X with the normal square X^2 and G_δ -diagonal and every function $g : X \rightarrow \mathbf{R}$ of the first Baire class there exists a separately continuous function $f : X \times X \rightarrow \mathbf{R}$, for which $f(x, x) = g(x)$, i.e. every the first Baire class function on the diagonal can be extended to a separately continuous function on all product. Analogous question for functions of n variables was considered in [3].

On other hand, in the investigations of separately continuous functions $f : X \times Y \rightarrow \mathbf{R}$ defined on the product of topological spaces X and Y the following two topologies naturally arise (see [4]): the separately continuous topology σ (the weakest topology with respect to which all functions f are continuous) and the cross-topology γ (it consists of all sets G for which all x -sections $G^x = \{y \in Y : (x, y) \in G\}$ and y -sections $G_y = \{x \in X : (x, y) \in G\}$ are open in Y and X respectively). Since the diagonal $\Delta = \{(x, x) : x \in \mathbf{R}\}$ is a closed discrete set in (\mathbf{R}^2, σ) or in (\mathbf{R}^2, γ) and not every function defined on Δ can be extended to a separately continuous function on \mathbf{R}^2 , even for $X = Y = \mathbf{R}$ the topologies σ and γ are not normal (moreover, γ is not regular [4,5]). Besides, every separately continuous function $f : X \times Y \rightarrow \mathbf{R}$ is a Baire class function for a wide class of the products $X \times Y$, in particular, if at least one of the multipliers is metrizable [6]. Thus, the following question naturally arises: for which sets $E \subseteq X \times Y$ and σ -continuous (γ -continuous) function $g : E \rightarrow \mathbf{R}$ of the first Baire class there exists a separately continuous function $f : X \times Y \rightarrow \mathbf{R}$ for which the restriction $f|_E$ coincides with g ?

In this paper we generalize an approach proposed in [2] and solve the problem formulated above for sets E of some type in the product of topological spaces.

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2. NOTIONS AND AUXILIARY STATEMENTS

A set $A \subseteq X$ has the extension property in a topological space X , if every continuous function $g : A \rightarrow [0, 1]$ can be extended to a continuous function $f : X \rightarrow [0, 1]$. According to Tietze-Uryson theorem [7, p.116], every closed set in a normal space has the extension property.

Lemma 2.1. *Let sets X_1 and Y_1 have the extension property in a topological spaces X and Y respectively, $e : X_1 \rightarrow Y_1$ be a homeomorphism, $E = \{(x, e(x)) : x \in X_1\}$ and $g : E \rightarrow [-1, 1]$ be a continuous function. Then there exist continuous functions $f : X \times Y \rightarrow [-1, 1]$ and $h : X \times Y \rightarrow [-1, 1]$, which satisfies the following conditions:*

- (i) $f|_E = g$;
- (ii) $E \subseteq h^{-1}(0)$;
- (iii) for every $x', x'' \in X$ and $y', y'' \in Y$ if $x' = x''$ or $y' = y''$ then $|f(x', y') - f(x'', y'')| = |h(x', y') - h(x'', y'')|$.

Proof. Consider the continuous function $\varphi : X_1 \rightarrow [-1, 1]$ and $\psi : Y_1 \rightarrow [-1, 1]$, which defined by: $\varphi(x) = g(x, e(x))$, $\psi(y) = g(e^{-1}(y), y)$. Since X_1 and Y_1 have the extension property in X and Y respectively, there exist continuous functions $\tilde{\varphi} : X \rightarrow [-1, 1]$ and $\tilde{\psi} : Y \rightarrow [-1, 1]$ such that $\tilde{\varphi}|_{X_1} = \varphi$ and $\tilde{\psi}|_{Y_1} = \psi$. Put $f(x, y) = \frac{\tilde{\varphi}(x) + \tilde{\psi}(y)}{2}$ and $h(x, y) = \frac{\tilde{\varphi}(x) - \tilde{\psi}(y)}{2}$. Clearly that f and h are continuous on $X \times Y$ and valued in $[-1, 1]$. Moreover, for every point $p = (x, y) \in E$ we have $\tilde{\varphi}(x) = \varphi(x) = g(p) = \psi(y) = \tilde{\psi}(y)$. Therefore $f|_E = g$ and $h|_E = 0$, i.e. the conditions (i) and (ii) are hold.

Let $x', x'' \in X$ and $y \in Y$. Then

$$f(x', y) - f(x'', y) = \frac{\tilde{\varphi}(x') - \tilde{\varphi}(x'')}{2} = h(x', y) - h(x'', y).$$

If $x \in X$ and $y', y'' \in Y$, then

$$f(x, y') - f(x, y'') = \frac{\tilde{\psi}(y') - \tilde{\psi}(y'')}{2} = h(x, y') - h(x, y'').$$

Thus, the condition (iii) is holds and lemma is proved. \square

In the case if the set E satisfies a compactness-type condition we will use the following proposition.

Lemma 2.2. *Let X be a topological space, E be a pseudocompact set in X , $(f_n)_{n=1}^\infty$ be a sequence of continuous functions $f_n : X \rightarrow \mathbf{R}$ which pointwise converges on the set E . Then there exists a functionally closed set $F \subseteq X$ such that $E \subseteq F$ and the sequence $(f_n)_{n=1}^\infty$ pointwise converges on the set F .*

Proof. Consider the diagonal mapping

$$f = \Delta_{n \in \mathbf{N}} f_n : X \rightarrow \mathbf{R}^{\mathbf{N}}, \quad f(x) = (f_n(x))_{n \in \mathbf{N}}.$$

Since the set E is pseudocompact and f is continuous, the set $f(E)$ is a pseudocompact set in the metrizable space $\mathbf{R}^{\mathbf{N}}$. Therefore $f(E)$ is closed and the set $F = f^{-1}(f(E))$ is functionally closed. It remains to verify that the sequence $(f_n)_{n=1}^\infty$ pointwise converges on F . Let $x \in F$. Then there exists an $x_1 \in E$ such that $f(x) = f(x_1)$, i.e. $f_n(x) = f_n(x_1)$ for every $n \in \mathbf{N}$. Since the sequence $(f_n(x_1))_{n=1}^\infty$ is convergent, the sequence $(f_n(x))_{n=1}^\infty$ is convergent too. \square

The following proposition we will use in a final stage of the construction of separately continuous functions with the given restriction.

Lemma 2.3. *Let X be a topological space, F be a functionally closed set in X , $(h_n)_{n=1}^\infty$ be a sequence of continuous functions $h_n : X \rightarrow \mathbf{R}$ such that $F \subseteq h_n^{-1}(0)$ for every $n \in \mathbf{N}$ $G = X \setminus F$. Then there exists a locally finite partition of the unit $(\varphi_n)_{n=0}^\infty$ on G such that the supports $G_n = \text{supp } \varphi_n = \{x \in G : \varphi_n(x) > 0\}$ of functions φ_n satisfy the conditions:*

- (a) $\overline{G_n} \cap F = \emptyset$ for every $n = 0, 1, 2, \dots$;
- (b) $G_n \subseteq h_n^{-1}((-\frac{1}{n}, \frac{1}{n}))$ for every $n = 1, 2, \dots$.

Proof. Let $h_0 : X \rightarrow [0, 1]$ be a continuous function such that $F = h_0^{-1}(0)$. For every $n \in \mathbf{N}$ we put

$$A_n = \bigcap_{k=0}^n h_k^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right), \quad B_n = \bigcap_{k=0}^n h_k^{-1}\left[-\frac{1}{n}, \frac{1}{n}\right],$$

$G_n = A_n \setminus B_{n+2}$ and, moreover, $G_0 = G \setminus B_2$. Clearly that all sets G_n are functionally open and $G_n \subseteq h_n^{-1}((-\frac{1}{n}, \frac{1}{n}))$ for every $n \in \mathbf{N}$, i.e. the condition (b) holds. Note that

$$\bigcap_{n=1}^\infty A_n = \bigcap_{n=1}^\infty B_n = \bigcap_{n=0}^\infty h_n^{-1}(0) = F.$$

Since $A_{n+1} \subseteq B_{n+1} \subseteq A_n$ for every $n \in \mathbf{N}$,

$$A_n \setminus A_{n+1} \subseteq A_n \setminus B_{n+2} \subseteq A_n \setminus A_{n+2} = (A_n \setminus A_{n+1}) \cup (A_{n+1} \setminus A_{n+2}).$$

Therefore

$$\bigcup_{n=1}^\infty G_n = \bigcup_{n=1}^\infty (A_n \setminus B_{n+2}) = \bigcup_{n=1}^\infty (A_n \setminus A_{n+1}) = A_1 \setminus \left(\bigcap_{n=1}^\infty A_n\right) = A_1 \setminus F.$$

Thus, $\bigcup_{n=0}^\infty G_n = (G \setminus B_2) \cup (A_1 \setminus F) = G$.

We show that the family $(G_n : n = 0, 1, \dots)$ is locally finite on G . Let $x \in G$, i.e. $h_0(x) \neq 0$. We choose $n_0 \in \mathbf{N}$ such that $\frac{1}{n_0} < |h_0(x)|$. Then $x \notin B_{n_0}$ and the set $G \setminus B_{n_0}$ is a neighborhood of x . On other hand, $G_n \subseteq A_n \subseteq B_{n_0}$ for every $n \geq n_0$. Therefore $G_n \cap (G \setminus B_{n_0}) = \emptyset$ for every $n \geq n_0$. Thus, the family $(G_n : n = 0, 1, \dots)$ is locally finite at the point x .

Since the sets G_n are functionally open, there exist continuous functions $\psi_n : X \rightarrow [0, 1]$ such that $G_n = \psi_n^{-1}((0, 1])$. The function $\psi : G \rightarrow [0, +\infty)$ which defined by $\psi(x) = \sum_{n=0}^\infty \psi_n(x)$ is continuous, moreover, $\text{supp } \psi = G$. For every $x \in G$

and $n = 0, 1, \dots$ we put $\varphi_n(x) = \frac{\psi_n(x)}{\psi(x)}$. The functions φ_n are continuous and formed a locally finite partition of the unit on G , moreover $G_n = \text{supp } \varphi_n$.

It remains to verify the condition (a). Since $G_n \subseteq X \setminus B_{n+2} \subseteq X \setminus A_{n+2}$ and the set $X \setminus A_{n+2}$ is closed, $\overline{G_n} \subseteq X \setminus A_{n+2}$, i.e. $\overline{G_n} \cap A_{n+2} = \emptyset$. Moreover, $F \subseteq A_{n+2}$, therefore $\overline{G_n} \cap F = \emptyset$ for every $n = 0, 1, \dots$ \square

3. MAIN RESULTS

Theorem 3.1. *Let sets X_1 and Y_1 have the extension property in topological spaces X and Y respectively, $e : X_1 \rightarrow Y_1$ be a homeomorphism, $E = \{(x, e(x)) : x \in X_1\}$, $g : E \rightarrow \mathbf{R}$ be the first Baire class function and at least one of the following conditions: E is pseudocompact, E is functionally closed in $X \times Y$, X_1 is functionally closed in X , Y_1 is functionally closed in Y holds. Then there exists a separately continuous function $f : X \times Y \rightarrow \mathbf{R}$ such that $f|_E = g$.*

Proof. We take a sequence of continuous functions $g_n : E \rightarrow [-n, n]$ which pointwise converges to the function g and use Lemma 2.1. We obtain a sequence of continuous functions $f_n : X \times Y \rightarrow [-n, n]$ and $h_n : X \times Y \rightarrow [-n, n]$ which satisfy the following conditions (i)-(iii).

We show that the set E is contained in some functionally closed set F_1 on which the sequence $(f_n)_{n=1}^\infty$ pointwise converges. If E is functionally closed, then $F_1 = E$. It follows from Lemma 2.2 the existence of such set F_1 for pseudocompact set E . It remains to verify this in the case when X_1 or Y_1 is functionally closed in X or Y respectively. Let X_1 is functionally closed in X . Now we put

$$F_1 = (X_1 \times Y) \cap \left(\bigcap_{n=1}^\infty h_n^{-1}(0) \right).$$

It follows from the property (ii) that E is contained in a functionally closed set F_1 . We take a point (x, y) from the set F_1 . Using the condition (iii) of Lemma 2.1 we obtain $|f_n(x, y) - f_n(x, e(x))| = |h_n(x, y) - h_n(x, e(x))| = 0$. Hence, $f_n(x, y) = f_n(x, e(x))$. Since the sequence $(f_n)_{n=1}^\infty$ pointwise converges on E , the sequence $(f_n(x, y))_{n=1}^\infty$ converges, because $(x, e(x)) \in E$.

Now we use Lemma 2.3 to the functionally closed set

$$F = F_1 \cap \left(\bigcap_{n=1}^\infty h_n^{-1}(0) \right)$$

in the space $X \times Y$ and to the sequence of continuous functions h_n and obtain a locally finite partition of the unit $(\varphi_n)_{n=0}^\infty$ on $G = (X \times Y) \setminus F$, which satisfies the conditions (a) and (b).

Let $f_0 \equiv 0$ on $X \times Y$. We consider the function

$$f(x, y) = \begin{cases} \sum_{n=0}^\infty \varphi_n(x, y) f_n(x, y), & \text{if } (x, y) \in G, \\ \lim_{n \rightarrow \infty} f_n(x, y), & \text{if } (x, y) \in F. \end{cases}$$

Since $(\varphi_n)_{n=0}^\infty$ is a locally finite partition of the unit on the set G and all functions f_n are continuous, the function f is correctly defined and continuous on the set G . Note that $F \subseteq F_1$. Therefore the sequence $(f_n)_{n=0}^\infty$ pointwise converges on F and the function f correctly defined on F . Moreover, since $E \subseteq h_n^{-1}(0)$ for every n and $E \subseteq F_1$, $E \subseteq F$ and $f|_E = \lim_{n \rightarrow \infty} f_n|_E = \lim_{n \rightarrow \infty} g_n = g$.

It remains to verify that the function f is separately continuous at points of the set F . Let $p_0 = (x_0, y_0) \in F$ $\varepsilon > 0$. We choose $n_0 \in \mathbf{N}$ such that $\frac{1}{n_0} < \frac{\varepsilon}{2}$ and $|f_n(p_0) - f(p_0)| < \frac{\varepsilon}{2}$ for every $n \geq n_0$. It follows from the condition (a) that the set

$$W = X \times Y \setminus \left(\bigcup_{n=0}^{n_0} \overline{G_n} \right),$$

where $G_n = \text{supp}\varphi_n$, is an open neighborhood of p_0 in $X \times Y$. We take a neighborhood U of x_0 in X such that $U \times \{y_0\} \subseteq W$. Let $x \in U$. If $p = (x, y_0) \in F$, then $h_n(p) = 0$ for every $n \in \mathbf{N}$. Then according to the condition (iii), we obtain $|f_n(p_0) - f_n(p)| = |h_n(p_0) - h_n(p)| = 0$, i.e. $f_n(p_0) = f_n(p)$. Therefore $f(p_0) = f(p)$. If $p \notin F$, then

$$f(p) = \sum_{n=0}^{\infty} \varphi_n(p) f_n(p) = \sum_{n=n_0}^{\infty} \varphi_n(p) f_n(p),$$

because $p \in W$. Then

$$\begin{aligned} |f(p_0) - f(p)| &= \left| \sum_{n=n_0}^{\infty} \varphi_n(p) (f(p_0) - f_n(p_0)) + \sum_{n=n_0}^{\infty} \varphi_n(p) f_n(p_0) - \right. \\ &\quad \left. - \sum_{n=n_0}^{\infty} \varphi_n(p) f_n(p) \right| \leq \sum_{n=n_0}^{\infty} \varphi_n(p) |f(p_0) - f_n(p_0)| + \\ &\quad + \sum_{n=n_0}^{\infty} \varphi_n(p) |f_n(p_0) - f_n(p)| < \sum_{n=n_0}^{\infty} \varphi_n(p) \cdot \frac{\varepsilon}{2} + \\ &\quad + \sum_{n=n_0}^{\infty} \varphi_n(p) |h_n(p_0) - h_n(p)| = \frac{\varepsilon}{2} + \sum_{n=n_0}^{\infty} \varphi_n(p) |h_n(p)|. \end{aligned}$$

It follows from the property (b) of sets G_n that if $\varphi_n(p) \neq 0$, then $|h_n(p)| < \frac{1}{n}$. Thus,

$$\sum_{n=n_0}^{\infty} \varphi_n(p) |h_n(p)| \leq \sum_{n=n_0}^{\infty} \varphi_n(p) \cdot \frac{1}{n} \leq \frac{1}{n_0} \sum_{n=n_0}^{\infty} \varphi_n(p) = \frac{1}{n_0} < \frac{\varepsilon}{2}.$$

Hence, $|f(p_0) - f(p)| < \varepsilon$. Thus, f is continuous at p_0 with respect to x .

The continuity of f at p_0 with respect to y can be proved analogously. Thus, f is separately continuous and the theorem is proved. \square

In the case of $X = Y = X_1 = Y_1$ we obtain the following theorem which generalized the result from [2].

Theorem 3.2. *Let X be a topological space and $g : X \rightarrow \mathbf{R}$ be a function of the first Baire class. Then there exists a separately continuous function $f : X \times X \rightarrow \mathbf{R}$ such that $f(x, x) = g(x)$ for every $x \in X$.*

4. FUNCTIONS ON THE PRODUCT OF COMPACTS

Now we consider the case when X and Y satisfy compactness type conditions. A set E in a product $X \times Y$ is called *horizontally and vertically onepointed* if for every $x \in X$ and $y \in Y$ the sets $E \cap (\{x\} \times Y)$ and $E \cap (X \times \{y\})$ are at most countable and *horizontally and vertically n -pointed* if corresponding sets contain at most n elements.

Theorem 4.1. *Let X and Y be compacts, E be a closed horizontally and vertically onepointed set in $X \times Y$ and $g : E \rightarrow \mathbf{R}$ be a function of the first Baire class. Then there exists a separately continuous function $f : X \times Y \rightarrow \mathbf{R}$, for which $f|_E = g$.*

Proof. Since the set E is horizontally and vertically onepointed, the projections the compact set E to the axis X and Y are continuous injective mappings. Hence, E is the graph of a homeomorphism $e : X_1 \rightarrow Y_1$, where X_1 and Y_1 are the projections of E on X and Y respectively. Now the existence of desired function f follows from Theorem 3.1. \square

Theorem 4.2. *Let X and Y be a locally compact spaces such that $X \times Y$ be a paracompact, E be a closed horizontally and vertically onepointed set and $g : E \rightarrow \mathbf{R}$ be the first Baire class function. Then there exists a separately continuous function $f : X \times Y \rightarrow \mathbf{R}$ for which $f|_E = g$.*

Proof. For every $p = (x, y) \in X \times Y$ we choose open neighborhoods U_p and V_p of x and y in X and Y respectively such that the closure $X_p = \overline{U_p}$ and $Y_p = \overline{V_p}$ are compacts and the set $E_p = E \cap (X_p \times Y_p)$ is horizontally and vertically onepointed. According to Theorem 4.2 there exists a separately continuous function $f_p : X_p \times Y_p \rightarrow \mathbf{R}$ for which $f_p|_{E_p} = g|_{E_p}$. Since the space $X \times Y$ is a paracompact, there exists a partition of the unit $(\varphi_i : i \in I)$ on $X \times Y$ which is subordinated to the open cover $(W_p = U_p \times V_p : p \in X \times Y)$ of $X \times Y$ [7, p.447]. For every $i \in I$ we choose $p_i \in X \times Y$ such that $\text{supp} \varphi_i \subseteq W_{p_i}$ and put

$$g_i(x, y) = \begin{cases} f_{p_i}(x, y), & \text{if } (x, y) \in W_{p_i}, \\ 0, & \text{if } (x, y) \notin W_{p_i}. \end{cases}$$

Note that the functions $\varphi_i \cdot g_i$ are separately continuous on $X \times Y$ and $(\varphi_i g_i)|_E = (\varphi_i|_E)g$. Then the function $f = \sum_{i \in I} \varphi_i g_i$ is the required. \square

5. EXAMPLE

Finally we give a example which show the essentiality of conditions under the set E in Theorem 4.1.

Let $X = Y = [0, 1]$,

$$E_1 = \left\{ \left(\frac{2k-1}{2^n}, \frac{2k-1}{2^n} + \frac{1}{2^{n+1}} \right) : k = 1, \dots, 2^{n-1}, n \in \mathbf{N} \right\},$$

$E_2 = \{(x, x) : x \in X\}$, $E = E_1 \cup E_2$, $g : E \rightarrow \mathbf{R}$, $g(x) = \begin{cases} 1, & x \in E_1, \\ 0, & x \in E_2. \end{cases}$ Clearly that E is a closed horizontally and vertically 2-pointed set in $X \times Y$ and g is a function of the first Baire class. Since the set E_1 is dense in E , the set $D(g)$ of discontinuity points set of the function g coincides with E . Therefore the projections of $D(g)$ on the axis X and Y coincide with X and Y respectively. On other hand, it is well-known that for every separately continuous function $f : X \times Y \rightarrow \mathbf{R}$ the set $D(f)$ of points of discontinuity of f is contained in the product $A \times B$ of meagre sets $A \subseteq X$ and $B \subseteq Y$ respectively. Thus, $D(g) \not\subseteq D(f)$ and the function f can not be extension of the function g .

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